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Ground state and low excitations of an integrable chain with alternating spins

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Abstract. An anisotropic integrable spin chain, consisting of spins $s = 1$ and $s = \frac{1}{2}$, is investigated [1]. It is characterized by two real parameters $\bar{c}$ and $\tilde{c}$, the coupling constants of the spin interactions. For the case $\bar{c} < 0$ and $\tilde{c} < 0$ the ground-state configuration is obtained by means of thermodynamic Bethe ansatz. Furthermore, the low excitations are calculated. It turns out that apart from free magnon states being the holes in the ground-state rapidity distribution, there exist bound states given by special string solutions of Bethe ansatz equations (BAE) in analogy to [13]. The dispersion law of these excitations is calculated numerically.

1. Introduction

Since the development of the quantum inverse scattering method (QISM) [2, 3] many generalizations of the well known $XXZ\left(\frac{1}{2}\right)$ model have been investigated [4–6]. The appearance of interesting new features of these models made these analyses very fruitful.

In 1992 de Vega and Woynarovich [1] suggested a new kind of generalization by constructing a model containing spin-$\frac{1}{2}$ and spin-1 particles. In comparison with models containing only one kind of spin it shows a richer structure due to additional parameters, the spin interaction coupling constants, making an investigation worthwhile.

Its isotropic limit $XXX\left(\frac{1}{2}, 1\right)$ has been studied in [14, 15]. The generalization procedure was extended to pairs of higher spins $\left(\frac{1}{2}, \mathrm{S}\right)$ and $(\mathrm{S}', \mathrm{S})$ in [16–18]. Except [14], only the conformally invariant case with positive coupling was considered. Our aim, therefore, is to go beyond that point. We intend to continue along this line in a subsequent publication.

In this paper we want to study the $XXZ\left(\frac{1}{2}, 1\right)$ model with strictly alternating spins. Definitions are reviewed in section 2. In section 3 we carry out the thermodynamic Bethe ansatz (TBA) and obtain the ground state for negative coupling of the spin interactions. Section 4 deals with the higher level Bethe ansatz for low excitations above this ground state and section 5 contains our conclusions.

2. Description of the model

We consider the Hamiltonian of a spin chain of length $2N$ [1]

$$\mathcal{H}(\gamma) = \bar{c}\bar{\mathcal{H}}(\gamma) + \tilde{c}\tilde{\mathcal{H}}(\gamma) - HS^z \tag{2.1}$$

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where $\bar{\mathcal{H}}$ couples two spins $s = \frac{1}{2}$ and one spin $s = 1$ while the converse is true for $\tilde{\mathcal{H}}$ (for explicit construction again see [1], bars and tildes are interchanged with respect to this reference as in subsequent publications of de Vega *et al*). H is an external magnetic field. We impose a periodic boundary condition.

The Hamiltonian contains an XXZ-type anisotropy, which is parametrized by $e^{i\gamma}$ or $e^{-\gamma}$, respectively. We investigate the weak antiferromagnetic case, i.e. the parametrization is $e^{i\gamma}$ and we restrict ourselves to $0 < \gamma < \pi/2$. Moreover, we have two additional real parameters $\tilde{c}$ and $\bar{c}$ being the coupling constants of the spin interactions and dominating the qualitative behaviour of the model.

The Bethe ansatz equations (BAE) determining the solution of the model are

$$\left(\frac{\sinh(\lambda_j + i\gamma/2)\sinh(\lambda_j + i\gamma)}{\sinh(\lambda_j - i\gamma/2)\sinh(\lambda_j - i\gamma)}\right)^N = -\prod_{k=1}^{M} \frac{\sinh(\lambda_j - \lambda_k + i\gamma)}{\sinh(\lambda_j - \lambda_k - i\gamma)} \qquad j = 1 \ldots M\,. \tag{2.2}$$

One can express energy, momentum and spin in terms of BAE roots λ_j:

$$E = \bar{c}\bar{E} + \tilde{c}\tilde{E} - \left(\frac{3N}{2} - M\right) H$$

$$\bar{E} = -\sum_{j=1}^{M} \frac{2\sin\gamma}{\cosh 2\lambda_j - \cos\gamma}$$

$$\tilde{E} = -\sum_{j=1}^{M} \frac{2\sin 2\gamma}{\cosh 2\lambda_j - \cos 2\gamma} \tag{2.3}$$

$$P = \frac{i}{2}\sum_{j=1}^{M}\left\{\log\left(\frac{\sinh(\lambda_j + i\gamma/2)}{\sinh(\lambda_j - i\gamma/2)}\right) + \log\left(\frac{\sinh(\lambda_j + i\gamma)}{\sinh(\lambda_j - i\gamma)}\right)\right\} \tag{2.4}$$

$$S_z = \frac{3N}{2} - M\,. \tag{2.5}$$

Here we have subtracted a constant in the momentum in order to make this magnitude vanishing for the ferromagnetic state. Furthermore, additive constants in $\bar{\mathcal{H}}$ and $\tilde{\mathcal{H}}$ are dropped due to different normalization compared with [1].

3. Thermodynamic Bethe ansatz (TBA) and ground state for negative coupling constants

We want to study the model in the thermodynamic limit $N \to \infty$. Then only solutions of the string-type occur (for M fixed):

$$\lambda_\alpha^{n,j,\nu} = \lambda_\alpha^{n,\nu} + i(n + 1 - 2j)\frac{\gamma}{2} + \frac{1}{4}i\pi(1-\nu) + \delta_\alpha^{n,j,\nu} \qquad j = 1 \ldots n\,. \tag{3.1}$$

Here $\lambda_\alpha^{n,\nu}$ is the real centre of the string, n is the string length and ν the parity of the string with values ± 1. The last term is a correction due to finite-size effects.

Now the permitted values of the string length n and related parities ν depending on the anisotropy γ are to be determined. Several approaches have been used to obtain them for the massive Thirring model [8] and for the $XXZ\left(\frac{1}{2}\right)$ [7] and $XXZ(S)$ models [10–12] all leading to the Takahashi conditions, a system of inequalities describing admissible string length and parities:

$$\nu_n \sin\gamma j \sin\gamma(n - j) > 0 \qquad j = 1 \ldots n - 1\,. \tag{3.2}$$

In the $XXZ(S)$ case additional restrictions on the spin arise.

By applying similar methods to our model one can show that fulfilling the Takahashi conditions is necessary and sufficient for an admissible pair (n, ν_n) for the number of magnons M to be fixed. On the other hand we did not succeed in deriving such conditions in the case $N, M \to \infty$, M/N fixed from the BAE directly. Nevertheless, there is a series of arguments favouring the conjecture that in this case the same strings as in the previous one can exist. In the following we want to state a few of them briefly.

Generally, the string hypothesis is believed to be valid in the case $N, M \to \infty$, M/N fixed (i.e. string corrections are exponentially small) for non-vanishing external magnetic field (see e.g. [9, 15]). For zero magnetic field the numerical results of Frahm *et al* show that string corrections are of orders $O(1/N)$ to $O(1)$, which does not affect the thermodynamics of the model.

For further discussions it is important that, as already proven in [1], the 'sea-strings' are exponentially exact.

Furthermore, Takahashi and Suzuki obtained a phenomenological rule determining admissible strings for the $XXZ\left(\frac{1}{2}\right)$ model proven to be equivalent to the Takahashi conditions. Fulfilling this criterion requires that a singlet state can be constructed by introducing a maximum number of strings of one type. Kirillov and Reshetikhin generalized this criterion naturally to higher spins. In the $XXZ(S)$ it is also a selection rule for permitted spin values as a criterion for an admissible pair (n, ν_n).

In the following we shall therefore assume the existence of strings fulfilling the Takahashi conditions.

Substituting (3.1) into (2.2) and taking the logarithm yields

$$Nt_{j,1}(\lambda_\alpha^{n_j}) + Nt_{j,2}(\lambda_\alpha^{n_j}) = 2\pi I_\alpha^{n_j} + \sum_k \sum_\beta \Theta_{jk}(\lambda_\alpha^{n_j} - \lambda_\beta^{n_k}, \nu_j\nu_k) \tag{3.3}$$

with the known notations

$$t_{j,2S}(\lambda) = \sum_{k=1}^{\min(n_j,2S)} f(\lambda, |n_j - 2S| + 2k - 1, \nu_j) \tag{3.4}$$

$$\Theta_{jk}(\lambda) = f(\lambda, |n_j - n_k|, \nu_j\nu_k) + f(\lambda, (n_j + n_k), \nu_j\nu_k) + 2\sum_{k=1}^{\min(n_j,n_k)-1} f(\lambda, |n_j - n_k| + 2k, \nu_j\nu_k) \tag{3.5}$$

and

$$f(\lambda, n, \nu) = \begin{cases} 0 & n\gamma/\pi \in \mathbf{Z} \\ 2\nu \arctan((\cot(n\gamma/2))^\nu \tanh\lambda) & n\gamma/\pi \notin \mathbf{Z}\,. \end{cases} \tag{3.6}$$

Here we have used the fact that a given string length $n > 1$ corresponds to a unique parity, which is a consequence of (3.2). The numbers $I_\alpha^{n_j}$ are half-odd-integers counting the strings of length n_j.

Introducing particle and hole densities in the usual way (see e.g. [5], equations (60)–(63), we perform the limiting process $N \to \infty$

$$a_{j,1}(\lambda) + a_{j,2}(\lambda) = (\rho_j(\lambda) + \tilde{\rho}_j(\lambda))(-1)^{r(j)} + \sum_k T_{jk} * \rho_k(\lambda) \tag{3.7}$$

where $a * b(\lambda)$ denotes the convolution

$$a * b(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}\mu\, a(\lambda - \mu) b(\mu) \tag{3.8}$$

and

$$a_{j,2S}(\lambda) = \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\lambda}t_{j,2S}(\lambda) \qquad T_{j,k}(\lambda) = \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\lambda}\Theta_{j,k}(\lambda)\,. \tag{3.9}$$

The sign $(-1)^{r(j)}$ results from the requirement of positive densities in the 'non-interacting' limit (i.e. only one string type is present) [12].

The analysis of (3.2) remains complicated for arbitrary γ [7], but at special values the picture becomes easy, namely $\gamma = \pi/\mu, \mu \ldots$ integer, $\mu \geqslant 3$. Following Babujian and Tsvelick [6] we want to consider this case only.

Due to periodicity we are left with strings

(i) $n_j = j$, $j = 1\ldots\mu-1$, $\nu_j = 1$,

(ii) $n_\mu = 1$, $\nu_\mu = -1$.

Equation (3.7) then reduces to

$$\begin{aligned} a_{j,1}(\lambda) + a_{j,2}(\lambda) &= \rho_j(\lambda) + \tilde{\rho}_j(\lambda) + \sum_{k=1}^{\mu} T_{jk} * \rho_k(\lambda) \qquad j = 1\ldots\mu-1 \\ a_{\mu,1}(\lambda) + a_{\mu,2}(\lambda) &= -\rho_\mu(\lambda) - \tilde{\rho}_\mu(\lambda) + \sum_{k=1}^{\mu} T_{\mu k} * \rho_k(\lambda)\,. \end{aligned} \tag{3.10}$$

We are now able to express energy, momentum and spin in terms of the densities via (2.3)–(2.5). The standard procedure leads to equations determining the equilibrium state at temperature T:

$$\begin{aligned} T\ln\left(1+\exp\left(\frac{\epsilon_j}{T}\right)\right) &= Hn_j - 2\pi\bar{c}a_{j,1}(\lambda) - 2\pi\tilde{c}a_{j,2}(\lambda) \\ &\quad + \sum_{k=1}^{\mu} T\ln\left(1+\exp\left(\frac{-\epsilon_k}{T}\right)\right) * A_{jk}(\lambda) \end{aligned} \tag{3.11}$$

with

$$A_{jk}(\lambda) = (-1)^{r(k)}T_{jk}(\lambda, \nu_j\nu_k) + \delta(\lambda)\delta_{jk} \tag{3.12}$$

and

$$\frac{\tilde{\rho}_j}{\rho_j} = \exp\left(\frac{\epsilon_j}{T}\right). \tag{3.13}$$

Again the free energy can be expressed in terms of our new variables $\epsilon_j(\lambda)$:

$$2\mathcal{F} = \frac{F}{N} = -\int_{-\infty}^{\infty}\mathrm{d}\lambda\,\sum_{j=1}^{\mu}(-1)^{r(j)}(a_{j,1}(\lambda)+a_{j,2}(\lambda))T\ln\left(1+\exp\left(\frac{-\epsilon_j}{T}\right)\right) - \frac{3H}{2}\,. \tag{3.14}$$

Reversing the operator A_{jk} in (3.11) by applying

$$C_{jk} = \delta(\lambda)\delta_{jk} - s(\lambda)(\delta_{j+1k} + \delta_{j-1k}) \qquad j,k = 1\ldots\mu \tag{3.15}$$

with

$$s(\lambda) = \frac{1}{2\gamma\cosh(\pi\lambda/\gamma)} \tag{3.16}$$

yields

$$
\begin{aligned}
&\epsilon_1(\lambda) = Ts * \ln(f(\epsilon_2)) - 2\pi\bar{c}s(\lambda)\\
&\epsilon_2(\lambda) = Ts * \ln(f(\epsilon_3)f(\epsilon_1)) - 2\pi\tilde{c}s(\lambda)\\
&\epsilon_j(\lambda) = Ts * \ln(f(\epsilon_{j+1})f(\epsilon_{j-1})) + \delta_{j\mu-2}Ts * \ln(f(-\epsilon_\mu)) \qquad j = 3\ldots\mu-2\\
&\epsilon_{\mu-1}(\lambda) = \frac{H\mu}{2} + Ts * \ln(f(\epsilon_{\mu-2}))\\
&\epsilon_\mu(\lambda) = \frac{H\mu}{2} - Ts * \ln(f(\epsilon_{\mu-2}))
\end{aligned}
\tag{3.17}
$$

with

$$
f(x) = 1 + \mathrm{e}^{x/T}\,. \tag{3.18}
$$

From these equations it can easily be established that $\epsilon_j \geqslant 0$ for $j = 3\ldots\mu-1$.

Since we are interested in the ground state we take the limit $T \to 0$ in equations (3.11) and (3.14). Taking into account that $\epsilon_j \geqslant 0$ for $j = 3\ldots\mu-1$ we have

$$
\epsilon_j^+(\lambda) = Hn_j - 2\pi\bar{c}a_{j,1}(\lambda) - 2\pi\tilde{c}a_{j,2}(\lambda) - \epsilon_1^- * A_{j1}(\lambda) - \epsilon_2^- * A_{j2}(\lambda) - \epsilon_\mu^- * A_{j\mu}(\lambda) \tag{3.19}
$$

$$
\begin{aligned}
2\mathcal{F} = \frac{F}{N} = \int_{-\infty}^{\infty} \mathrm{d}\lambda\, \big[&(a_{1,1}(\lambda) + a_{1,2}(\lambda))\epsilon_1^-(\lambda) + (a_{2,1}(\lambda) + a_{2,2}(\lambda))\epsilon_2^-(\lambda)\\
&-(a_{\mu,1}(\lambda)) + a_{\mu,2}(\lambda))\epsilon_\mu^-(\lambda)\big] - \frac{3H}{2}
\end{aligned}
\tag{3.20}
$$

where ϵ_j^+ and ϵ_j^- denote positive and negative parts of the function ϵ_j, respectively.

Now we want to discuss equation (3.19) for the case $H = 0^+$ and its dependence on the signs of $\bar{c}$ and $\tilde{c}$.

(i) $\bar{c} > 0$, $\tilde{c} > 0$. This sector has been investigated completely by de Vega and Woynarovich [1]. The solution is

$$
\epsilon_j(\lambda) = -2\pi\bar{c}s(\lambda)\delta_{j1} - 2\pi\tilde{c}s(\lambda)\delta_{j2}\,. \tag{3.21}
$$

(ii) $\bar{c} > 0$, $\tilde{c} < 0$. The picture at this sector is difficult, since the quadrant is divided by a phase line starting in the origin and going to infinity, which we obtained from the TBA equations. In the lower area the model behaves as in case (iv), while in the upper one we expect for the ground state a mixture of strings of length 1 with different parities and finite Fermi zones. We hope to return to this case in greater detail later.

(iii) $\bar{c} < 0$, $\tilde{c} > 0$. The situation in this sector is similar to the case above. Again we have a phase line producing one area of case (iv) type and another one where the ground state is expected to be formed by the $(2,+)$- and $(1,-)$-strings again having finite Fermi zones.

(iv) $\bar{c} \leqslant 0$, $\tilde{c} \leqslant 0$. In this case $\epsilon_1(\lambda)$ and $\epsilon_2(\lambda)$ are positive. The only string present in the ground state is $(1,-)$. Equations (3.19) and (3.20) reduce to

$$
\epsilon_j^+(\lambda) = -2\pi\bar{c}a_{j,1}(\lambda) - 2\pi\tilde{c}a_{j,2}(\lambda) - \epsilon_\mu^- * A_{j\mu}(\lambda) \tag{3.22}
$$

$$
2\mathcal{F} = \frac{F}{N} = -\int_{-\infty}^{\infty} \mathrm{d}\lambda\, (a_{\mu,1}(\lambda) + a_{\mu,2}(\lambda))\epsilon_\mu^-(\lambda)\,. \tag{3.23}
$$

The solutions of (3.22) can be given via their Fourier transforms:

$$\begin{aligned}
\hat{\epsilon}_1(p) &= -2\pi\bar{c}\frac{\cosh[p\gamma(\mu-2)/2]}{\cosh[p\gamma(\mu-1)/2]} - 2\pi\tilde{c}\frac{\cosh[p\gamma(\mu-3)/2]}{\cosh[p\gamma(\mu-1)/2]} \\
\hat{\epsilon}_j(p) &= -2\pi\bar{c}\frac{\cosh[p\gamma(\mu-j-1)/2]}{\cosh[p\gamma(\mu-1)/2]} - 2\pi\tilde{c}\frac{2\cosh(p\gamma/2)\cosh[p\gamma(\mu-j-1)/2]}{\cosh[p\gamma(\mu-1)/2]} \\
\hat{\epsilon}_{\mu-1}(p) &= -2\pi\bar{c}\frac{1}{2\cosh[p\gamma(\mu-1)/2]} - 2\pi\tilde{c}\frac{\cosh(p\gamma/2)}{\cosh[p\gamma(\mu-1)/2]} \\
\hat{\epsilon}_\mu(p) &= 2\pi\bar{c}\frac{1}{2\cosh[p\gamma(\mu-1)/2]} + 2\pi\tilde{c}\frac{\cosh(p\gamma/2)}{\cosh[p\gamma(\mu-1)/2]}\,.
\end{aligned} \tag{3.24}$$

The ground energy state is

$$\begin{aligned}
2\mathcal{F} = \frac{F}{N} &= \bar{c}\int_{-\infty}^{\infty} \mathrm{d}p\, \frac{\sinh(p\gamma/2)+\sinh(p\gamma)}{2\cosh[p\gamma(\mu-1)/2]\sinh(p\gamma\mu/2)} \\
&\quad + \tilde{c}\int_{-\infty}^{\infty} \mathrm{d}p\, \frac{[\sinh(p\gamma/2)+\sinh(p\gamma)]\cosh(p\gamma/2)}{\cosh[p\gamma(\mu-1)/2]\sinh(p\gamma\mu/2)}\,.
\end{aligned} \tag{3.25}$$

For $\bar{c} = \tilde{c}$ the continuum limit provides a conformal invariant theory. Due to the existence of *one* type of elementary excitations the central charge of the Virasoro algebra is $c = 1$. That is remarkable, because in section (i) $c = 2$ [1]. The point $\bar{c} = \tilde{c} = 0$ must be singular then, when passed on the conformal line.

(v) $\bar{c} = 0$, $\tilde{c} > 0$. (vi) $\tilde{c} > 0$, $\bar{c} = 0$. Both cases have been briefly considered in [14]. The two lines are infinitely high singular, because of the infinite degeneration of the ground state.

In the following we shall consider case (iv) only.

4. Higher level Bethe ansatz for low excitations

In this section we want to derive equations for excitations above the ground state in the case $\bar{c} < 0$, $\tilde{c} < 0$. The starting point of our analysis is the result of section 3. Though we have derived the ground-state configuration of BAE roots only for the special case $\gamma = \pi/\mu$, μ integer, we extend this result to the whole range of γ, $0 < \gamma < \pi/2$, motivated by the results of Frahm *et al* [12], who found for the $XXZ(S)$ Hamiltonians, where only special intervals of the anisotropy γ are permitted, that the ground-state configuration does not change within any of these intervals. Since in our case no restrictions on γ arise, i.e. we have only one permitted interval, we expect the ground-state configuration to be the one obtained for the above-mentioned special values in the whole γ-range.

We write down the BAE for this ground state in the limit $N \to \infty$ with a finite number of excitations (holes in the ground-state rapidity distribution and additional complex roots):

$$\begin{aligned}
\bar{\phi}'(\lambda,\gamma/2) + \bar{\phi}'(\lambda,\gamma) &= -\rho(\lambda) - \frac{1}{N}\sum_{h=1}^{N_h}\delta(\lambda-\lambda_h) + \int_{-\infty}^{\infty}\mathrm{d}\lambda'\,\rho(\lambda')\phi'(\lambda-\lambda',\gamma) \\
&\quad + \frac{1}{N}\sum_{l=1}^{N_l}\bar{\phi}'(\lambda-z_l,\gamma)\,.
\end{aligned} \tag{4.1}$$

Here we have introduced the new notations

$$\phi\left(\lambda,\frac{n\gamma}{2}\right) = \frac{1}{2\pi}f(\lambda,n,+1) \qquad \bar{\phi}\left(\lambda,\frac{n\gamma}{2}\right) = \frac{1}{2\pi}f(\lambda,n,-1) \tag{4.2}$$

and the prime means the derivative with respect to the first argument. The numbers of holes λ_h and additional complex roots z_l are denoted by N_h and N_l, respectively. The hole positions are defined as solutions of (3.3) for the omitted I_h.

The solution of (4.1) contains different contributions obtained by Fourier transformation,

$$\rho(\lambda) = \rho_0(\lambda) + \frac{1}{N}(\rho_h(\lambda) + \rho_c(\lambda) + \rho_w(\lambda)) \tag{4.3}$$

with

$$\begin{aligned}
\hat{\rho}_0(p) &= \frac{1 + 2\cosh(p\gamma/2)}{2\cosh(p(\pi-\gamma)/2)} \\
\hat{\rho}_h(p) &= -\sum_{h=1}^{N_h} \frac{\mathrm{e}^{-\mathrm{i}p\lambda_h}\sinh(p\pi/2)}{2\sinh(p\gamma/2)\cosh(p(\pi-\gamma)/2)} \\
\hat{\rho}_c(p) &= -\sum_{l=1}^{N_c/2} \mathrm{e}^{-\mathrm{i}p\sigma_l} \frac{\cosh(p\gamma/2)\cosh(p\tau_l)}{\cosh(p(\pi-\gamma)/2)} \\
\hat{\rho}_w(p) &= -\sum_{l=1}^{N_w/2} \mathrm{e}^{-\mathrm{i}p\sigma_l} \frac{\sinh(p(\pi-\gamma)/2)\cosh(p(\pi-2\tau_l)/2)}{\cosh(p(\pi-\gamma)/2)\sinh(p\gamma/2)}\,.
\end{aligned} \tag{4.4}$$

Here we have distinguished between complex close ($|\,\mathrm{Im}(z_l)| < \pi/2 - \gamma$) and wide roots ($|\,\mathrm{Im}(z_l)| > \pi/2 - \gamma$) differing in their Fourier transforms. Moreover, we have used the fact that complex roots appear in conjugated pairs $(z_l, \bar{z}_l)$ only.

The energy and momentum of the elementary hole excitations are given through

$$\begin{aligned}
\varepsilon_h(\lambda_h) &= \bar{\varepsilon}_h(\lambda_h) + \tilde{\varepsilon}_h(\lambda_h) \\
\bar{\varepsilon}_h(\lambda_h) &= -\frac{\pi\bar{c}}{\pi-\gamma}\,\frac{1}{\cosh(\pi\lambda_h/(\pi-\gamma))} \\
\tilde{\varepsilon}_h(\lambda_h) &= -\frac{4\pi\tilde{c}}{\pi-\gamma}\,\frac{\cos(\pi\gamma/2(\pi-\gamma))\cosh(\pi\lambda_h/(\pi-\gamma))}{\cosh(2\pi\lambda_h/(\pi-\gamma)) + \cos(\pi\gamma/(\pi-\gamma))}
\end{aligned} \tag{4.5}$$

$$p_h(\lambda_h) = \frac{1}{2}\arctan\left(\sinh\left(\frac{\pi\lambda_h}{\pi-\gamma}\right)\right) + \frac{\pi}{4} + \arctan\left(\frac{\sinh(\pi\lambda_h/(\pi-\gamma))}{\cos(\pi\gamma/2(\pi-\gamma))}\right) + \frac{\pi}{2}\,. \tag{4.6}$$

We did not find an explicit expression for the dispersion law of hole excitations. Nevertheless, numerical calculation suggests that the curve shows the expected behaviour (cf figure 2).

Now we want to derive equations for hole positions and complex roots excluding the vacuum parameters by using the method of Babelon *et al* [13]. For this purpose we write down the BAE in integral approximation for a complex root z:

$$\exp(2\pi\mathrm{i}[I(z) - F(z)]) = -1 \tag{4.7}$$

with

$$I'(z) = N\int_{-\infty}^{\infty} \mathrm{d}\lambda'\,\sigma(\lambda')\bar{\phi}'(z-\lambda',\gamma) \tag{4.8}$$

$$F'(z) = N\phi'\left(z, \tfrac{1}{2}\gamma\right) + N\phi'(z,\gamma) + \sum_{h=1}^{N_h}\bar{\phi}'(z-\lambda_h,\gamma) + \sum_{l=1}^{N_c}\phi'(z-z_l,\gamma) \tag{4.9}$$

and $\sigma(\lambda)$ is the regular density

$$\sigma(\lambda) = \rho(\lambda) + \frac{1}{N}\sum_{h=1}^{N_h}\delta(\lambda-\lambda_h)\,. \tag{4.10}$$

$I'(z)$ has discontinuities on the lines $\mathrm{Im}(z) = \pm(\pi/2 - \gamma)$. So analysing (4.7) we have to consider the following three cases.

(a) $\mathrm{Im}(z_l) > \pi/2 - \gamma$. We can evaluate the function $I(z) - F(z)$ directly by integrating

$$I'(z) - F'(z) = -\sigma\left(z - \frac{\mathrm{i}\pi}{2}\right)N\,. \tag{4.11}$$

Noticing

$$I(\infty) - F(\infty) = 0 \tag{4.12}$$

one gets

$$I(z) - F(z) = -N\int_\infty^z \sigma\left(u - \frac{\mathrm{i}\pi}{2}\right)\mathrm{d}u \tag{4.13}$$

and

$$\exp\left(-2\pi\mathrm{i}N\int_\infty^z \sigma\left(u - \frac{\mathrm{i}\pi}{2}\right)\mathrm{d}u\right) = -1\,. \tag{4.14}$$

Splitting $\sigma(u)$ into vacuum and excitation contributions

$$\sigma\left(u - \frac{\mathrm{i}\pi}{2}\right) = \sigma_0\left(u - \frac{\mathrm{i}\pi}{2}\right) + \frac{1}{N}\bar{\sigma}\left(u - \frac{\mathrm{i}\pi}{2}\right) \tag{4.15}$$

we get for (4.14)

$$\exp\left(2\pi\mathrm{i}N\int_\infty^z \sigma_0\left(u - \frac{\mathrm{i}\pi}{2}\right)\mathrm{d}u + 2\pi\mathrm{i}\int_\infty^z \bar{\sigma}\left(u - \frac{\mathrm{i}\pi}{2}\right)\mathrm{d}u\right) = -1\,. \tag{4.16}$$

The first exponent term contains an explicit N dependence, while the second does not. So the validity of the above equation in the limit $N \to \infty$ requires, that the second term cancels the N dependence in the real part of the first one by approaching z in exponential order towards a pole of the integrand. These poles of $I'(u) - F'(u)$ are

$$z_l \pm \mathrm{i}\gamma \qquad \lambda_h \pm \left(\frac{\mathrm{i}\pi}{2} - \mathrm{i}\gamma\right) \qquad \pm\frac{1}{2}\mathrm{i}\gamma \qquad \pm\mathrm{i}\gamma \tag{4.17}$$

with the relevant ones being

$$z_l \pm \mathrm{i}\gamma \qquad \text{with residue } \mp\mathrm{i}\,. \tag{4.18}$$

Now it can be easily established that

$$\mathrm{Im}\left(\int_\infty^z \sigma_0\left(u - \frac{\mathrm{i}\pi}{2}\right)\mathrm{d}u\right) > 0\,. \tag{4.19}$$

Therefore it must exist z_l with $z = z_l + \mathrm{i}\gamma$.

(b) $-\pi/2 + \gamma < \mathrm{Im}(z_l) < \pi/2 - \gamma$. Here evaluating (4.8) by deforming the integration contour provides an additional term due to residue theorem. One has to replace

$$I(z) \to I(z) - N\sigma\left(z - \frac{\mathrm{i}\pi}{2} + \mathrm{i}\gamma\right). \tag{4.20}$$

Then equation (4.7) reads as

$$\exp\left(2\pi\mathrm{i}N\int_\infty^z \left[\sigma\left(u - \frac{\mathrm{i}\pi}{2}\right) - \sigma\left(z - \frac{\mathrm{i}\pi}{2} + \mathrm{i}\gamma\right)\right]\mathrm{d}u\right) = -1\,. \tag{4.21}$$

Using the Fourier transforms (4.4) one can establish the relation

$$\sigma_0(u) + \sigma_0(u - \mathrm{i}\pi + \mathrm{i}\gamma) = 0\,. \tag{4.22}$$

Then determining the sign of the real part of the exponent leads to

$$\mathrm{Im}\left(\int_{\infty}^{z}\left[\sigma_0\left(u+\frac{\mathrm{i}\pi}{2}\right)+\sigma_0\left(u-\frac{\mathrm{i}\pi}{2}\right)\right]\mathrm{d}u\right) \qquad \begin{cases} > 0 : \mathrm{Im}(z) > 0 \\ > 0 : \mathrm{Im}(z) < 0 \end{cases} \tag{4.23}$$

and we conclude that there exists z_l with $z = z_l + \mathrm{i}\gamma$ for $\mathrm{Im}(z) > 0$ and $z = z_l - \mathrm{i}\gamma$ for $\mathrm{Im}(z) < 0$, respectively.

(c) $\mathrm{Im}(z_l) < -\pi/2 + \gamma$. Evaluating (4.8) again by contour deformation and substituting into (4.7) leads to

$$\exp\left(2\pi\mathrm{i}N\int_{\infty}^{z}\left[\sigma\left(u-\frac{\mathrm{i}\pi}{2}\right)-\sigma\left(z-\frac{\mathrm{i}\pi}{2}+\mathrm{i}\gamma\right)+\sigma\left(z+\frac{\mathrm{i}\pi}{2}-\mathrm{i}\gamma\right)\right]\mathrm{d}u\right) = -1\,. \tag{4.24}$$

Using equation (4.22) the N-dependent exponent term reduces to

$$2\pi\mathrm{i}N\int_{\infty}^{z}\sigma_0\left(u+\frac{\mathrm{i}\pi}{2}\right)\mathrm{d}u\,. \tag{4.25}$$

With

$$\mathrm{Im}\left(\int_{\infty}^{z}\sigma_0\left(u+\frac{\mathrm{i}\pi}{2}\right)\mathrm{d}u\right) < 0 \tag{4.26}$$

it follows that $z = z_l - \mathrm{i}\gamma$, where z_l is a BAE root.

Now we are able to determine the configurations in which complex roots can appear. We distinguish three cases for the anisotropy:

(a) $0 < \gamma < \pi/4$. Because of symmetry with respect to the real axis we only consider $\mathrm{Im}(z) > 0$. This leads to the existence of z_l with $z = z_l + \mathrm{i}\gamma$. Repeating this argument for $z_l(\mathrm{Im}(z_l) > 0$ because of $\gamma < \pi/4)$ generates another root $z_l - \mathrm{i}\gamma$. This process of generating roots breaks down if the last generated root has a negative imaginary part. Therefore the 'minimal' configuration induced by z is a chain with spacing $\mathrm{i}\gamma$ situated completely in the upper half plane except for the last generated root.

For further discussion one has to distinguish wide and close roots.

(i) $\mathrm{Im}(z) > \pi/2 - \gamma$. Taking the product over equations (4.14) and (4.21), respectively, for all chain members shows that there remains a non-compensated N-dependent term until the chain is continued into the lower region of wide roots. Then the configuration becomes 'stable' in the limit $N \to \infty$. Notice that also the configuration with all roots complex conjugated exists, so that the configuration is actually a double chain.

(ii) $0 < \mathrm{Im}(z) < \pi/2 - \gamma$. Here the above procedure does not lead to a 'stable' configuration. The only way for this configuration to exist in the limit $N \to \infty$ is arranging the roots symmetrically with respect to the real axis. Therefore this configuration is of the string type. It coincides with its complex conjugated one (see figure 1).

(b) $\pi/4 < \gamma < \pi/3$ and (c) $\pi/3 < \gamma < \pi/2$. Analogous arguments again lead to short double chains and strings (see figure 1).

We can write down the possible configurations for the three cases in a more compact way:

(i) $z_t = z - \mathrm{i}t\gamma, t = 0\ldots n, n = [\pi/\gamma], [\pi/\gamma] - 1, z$ is a wide root

(ii) $z_t = z - \mathrm{i}t\gamma, t = 0\ldots n, n \leqslant n_{\mathrm{max}}, n_{\mathrm{max}} = [\pi/\gamma] - 2, \mathrm{Im}(z) = n\gamma/2, z$ is a close root.

The higher level BAE for a state with complex root configurations of the first type can be obtained by taking the product over equations (4.7) for all members and calculating

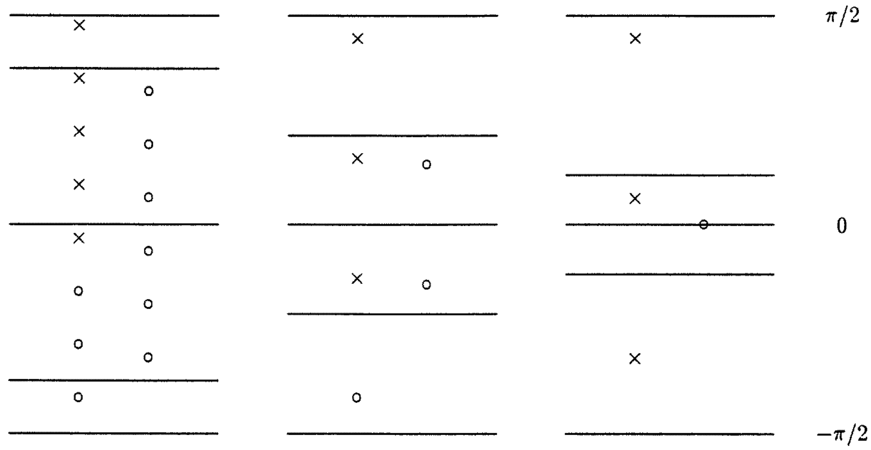

Figure 1. Some special configurations of complex roots for the three cases of γ. A member of the 'minimal' configuration is symbolized by '$\times$' and 'o' denotes additional roots to make the configuration 'stable'. String configuration members are also denoted by 'o', due to the fact that stability is realized by symmetrical arrangement. The lines varying in the three cases denotes $\pm(\pi/2-\gamma)$. Notice that for non-symmetric configurations the complex conjugated one in this picture is omitted.

$I(z)-F(z)$ directly via Fourier transforms. With the new parameters

$$\begin{aligned}
&\chi_l = z_l - \frac{\mathrm{i}n_l\gamma}{2} \qquad \text{for } n \text{ even}\\
&\tilde{\chi}_l = z_l - \frac{\mathrm{i}n_l\gamma}{2} \qquad \text{for } n \text{ odd}\\
&P \ldots \text{number of } \chi\text{-configurations}\\
&\tilde{P} \ldots \text{number of } \tilde{\chi}\text{-configurations}\\
&\alpha = \frac{\pi}{\pi-\gamma}
\end{aligned} \tag{4.27}$$

it follows that

$$\begin{aligned}
&\prod_{h=1}^{N_h} \frac{\sinh(\alpha(\chi_l-\lambda_h)+\mathrm{i}\alpha\pi/2)}{\sinh(\alpha(\chi_l-\lambda_h)-\mathrm{i}\alpha\pi/2)}\\
&\qquad = -\prod_{j=1}^{P} \frac{\sinh(\alpha(\chi_l-\chi_j)+\mathrm{i}\alpha\pi)}{\sinh(\alpha(\chi_l-\chi_j)-\mathrm{i}\alpha\pi)} \prod_{j=1}^{\tilde{P}} \frac{\cosh(\alpha(\chi_l-\tilde{\chi}_j)+\mathrm{i}\alpha\pi)}{\cosh(\alpha(\chi_l-\tilde{\chi}_j)-\mathrm{i}\alpha\pi)}\\
&\prod_{h=1}^{N_h} \frac{\cosh(\alpha(\tilde{\chi}_l-\lambda_h)+\mathrm{i}\alpha\pi/2)}{\cosh(\alpha(\tilde{\chi}_l-\lambda_h)-\mathrm{i}\alpha\pi/2)}\\
&\qquad = -\prod_{j=1}^{P} \frac{\cosh(\alpha(\tilde{\chi}_l-\chi_j)+\mathrm{i}\alpha\pi)}{\cosh(\alpha(\tilde{\chi}_l-\chi_j)-\mathrm{i}\alpha\pi)} \prod_{j=1}^{\tilde{P}} \frac{\sinh(\alpha(\tilde{\chi}_l-\tilde{\chi}_j)+\mathrm{i}\alpha\pi)}{\sinh(\alpha(\tilde{\chi}_l-\tilde{\chi}_j)-\mathrm{i}\alpha\pi)}\,.
\end{aligned} \tag{4.28}$$

Calculating energy and momentum for the configurations shows that for the first type the direct contribution to these magnitudes vanishes. On the other hand, there are two hole parameters associated with such a configuration for symmetrical arrangement ('string type') and four for a non-symmetric configuration ('double chain'). So these configurations do carry energy, but only via their associated holes. This is the standard picture, which also holds in our model.

For the second type we get

$$\Delta E = -\bar{c}\frac{4\pi}{\pi-\gamma}\frac{\sin[(n+1)\gamma\pi/2(\pi-\gamma)]\cosh[\sigma\pi/(\pi-\gamma)]}{\cosh[2\sigma\pi/(\pi-\gamma)]-\cos[(n+1)\gamma\pi/(\pi-\gamma)]}$$
$$-\tilde{c}\frac{4\pi}{\pi-\gamma}\left\{\frac{\sin[(n+2)\gamma\pi/2(\pi-\gamma)]\cosh[\sigma\pi/(\pi-\gamma)]}{\cosh[2\sigma\pi/(\pi-\gamma)]-\cos[(n+2)\gamma\pi/(\pi-\gamma)]}\right.$$
$$\left.+\frac{\sin[n\gamma\pi/2(\pi-\gamma)]\cosh[\sigma\pi/(\pi-\gamma)]}{\cosh[2\sigma\pi/(\pi-\gamma)]-\cos[n\gamma\pi/(\pi-\gamma)]}\right\} \tag{4.29}$$

$$\Delta P = \sum_{i=0}^{2}\arctan\left(\frac{\sinh[\sigma\pi/(\pi-\gamma)]}{\sin[(n+\mathrm{i})\gamma\pi/2(\pi-\gamma)]}\right)+\frac{3\pi}{2} \tag{4.30}$$

for $n \geqslant 1$. The dispersion law for these excitations has been calculated numerically for different $\gamma < \pi/3$ and string length n and $\bar{c}/\tilde{c} = 1$. A comparison with the dispersion law of a free two-magnon state, where the momenta of the magnons are equal as given in figure 2, reveals that these excitations can be identified with bound magnon states in analogy to [13]. One can see from the picture, that the maximum of the curve decreases monotonically with increasing n up to $n_{\max}$ for γ fixed, approaching the dispersion curve for the free state. On the other hand for $\gamma \to \pi/3$, where only the $n = 1$ state exists, again the bound-state curve approaches the one for the free state monotonically.

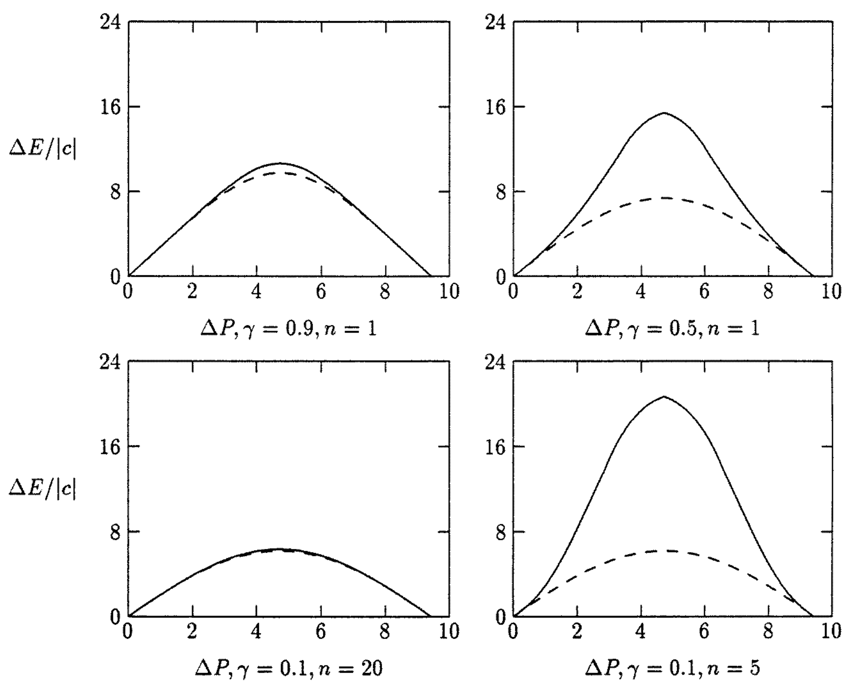

Figure 2. Dispersion relation for type II excitations (full curves) compared with the dispersion law for a free two-magnon state (broken curves) for different γ.

Moreover, there is a state with $n = 0$ allowed through the whole γ-range. Its energy and momentum are given via (4.29) and (4.30), where in the last formula the term for $i = 0$ is dropped. This leads to a range for the momentum between those for a single hole excitation and a bound state. Therefore the bound-state interpretation fails.

5. Conclusions

We have investigated a generalized Heisenberg spin chain with alternating spins $XXZ\left(\frac{1}{2}, 1\right)$ in the gapless region. By means of thermodynamic Bethe ansatz (TBA) integral equations determining the ground state have been derived. In the case of negative coupling of spin interactions these equations were solved. It turns out that the ground state is formed by a sea of $(1, -)$-strings and is therefore of antiferromagnetic nature.

Weakly excited states above this antiferromagnetic vacuum have been analysed following the method introduced in [13]. In analogy with the $XXZ\left(\frac{1}{2}\right)$ model for $-1 < \Delta < 0$ two types of excitations appear. The first one are scattering states of magnons. Higher level BAE are derived which determine the parameters of these excitations. The second type can be identified with bound magnon states in analogy to [13].

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